DISPERSIVE GRID-FREE ORTHOGONAL MATCHING PURSUIT FOR MODAL ESTIMATION IN OCEAN ACOUSTICS

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ABSTRACT

Considering low-frequency acoustic sources, shallow-water environments act as modal dispersive waveguides. In this context, the signal can be described as a sum of a few modal components, each of them propagating with its own wavenumber. When dealing with broadband sources, wavenumber-frequency (f-k) diagrams constitute popular representations naturally enabling modal separation. Based on a Fourier transform, they require however a large number of sensors to resolve wavenumbers with a high-resolution. This limitation can be overcame by adding some physical priors to the processing method. In the continuation of previous works, we propose here a new grid-free algorithm allowing a super-resolution of the (f-k) diagram by benifiting from the sparse nature of the wavenumber spectrum and embedding the broadband behavior of the wavenumbers within the algorithm. The method is validated on simulated data.

***Index Terms*—** Wavenumbers estimation, sparse representation, grid-free algorithms, greedy algorithms

1. INTRODUCTION

In shallow-water environments, propagation of acoustic low frequency signals is ruled by the modal theory: it can be described as the sum of few dispersive modes, each being defined by its own wavenumber and amplitude. Estimating them is of crucial importance to understand the propagating environment as well as the emitting source [1]. In particular, modes parameters are involved in matched mode processing [2] that constitutes a common approach to infer the properties of the environment [3]. The method consists firstly of decomposing the measured fields to obtain the excitations, that is, amplitudes of the constituent propagating modes. Then the source location can be estimated by matching these excitations with modeled replica excitations. A common approach to recover modes is the use of Fourier transform (FT). Indeed, as long as the acoustic field can be sampled in the range dimension, the mode estimation procedure is equivalent to a spectral estimation problem. As a result, considering a long horizontal line array (HLA) and a monochromatic source at the endfire position, the wavenumber spectrum can be obtained by applying a Fourier transform in the array dimension [4, 5]. When considering a broadband source, spatial wavenumber spectra can be estimated for each of the source frequencies. Stacking these spectra results in a frequency-wavenumber (f-k) diagram representative of the waveguide dispersion [4, 5, 6]. A disadvantage of the standard discrete Fourier transform is that it assumes regular sampling, resulting in low resolution of spatial wavenumbers when the HLA has few sensors. The resulting (f-k) diagram is then of poor quality. To overcome this problem, we propose to account for the sparsity of the contribution of modes in the spatial spectra. In addition, based on ideas introduced in previous works [7, 8], which explicit prior information on both the number of modes expected to propagate at each frequency and their links from one frequency to another, we address mode tracking among spatial spectra at successive time frequencies. Our contribution goes beyond these works : in addition to taking

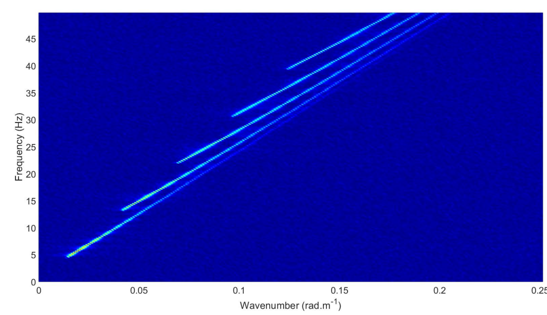

Fig. 1. (f-k) diagram based on data obtained by a Pekeris guide

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into account physical priors, we propose an algorithm that is free of the constraint of grid search for the wavenumber estimation. The paper is organized as follows. The second section recalls physical principles of modal propagation that will be exploited as prior knowledge to build up the (f-k) diagram. The third section is dedicated to the explanation of the grid-free utility and the proposed algorithm. Finally, in section 4, we present metrics adapted to our grid-free problem and the method is applied to synthetic simulations.

2. ACOUSTIC PROPAGATION IN DISPERSIVE SHALLOW WATER ENVIRONMENTS

When considering an emitting source $s(f)$ at depth z_s and frequency f, the received signal on a sensor located at the distance r and depth z can be written as [9] :

$$y(f,r,z,z_s) \simeq QS(f) \sum_{m=1}^{M(f)} A_m(f,z,z_s) e^{-jrk_{rm}(f)}, \quad (1)$$

where Q is a constant factor, $M(f)$ is the number of propagating modes at frequency f, $k_{rm}(f)$ (resp. $A_m(f,z,z_s)$) is the horizontal wavenumber (resp. modal amplitude) of the m^{th} mode. In modal theory, the horizontal wavenumbers associated to the propagating modes are linked to their vertical counter-parts by the dispersion relation, that is, for a given f,

$$\left(\frac{2\pi f}{c}\right)^2 = k_{rm}(f)^2 + k_{zm}(f)^2, \quad (2)$$

with c the sound speed in the medium. Discretizing the frequency axis (with $f = \nu\Delta_f$, $\nu \in \{0, ..., F\}$, Δ_f the frequency spacing) and denoting $k_{rm}[\nu] \triangleq k_{rm}(\nu\Delta_f)$, the wavenumbers attached to two successive indices are linked as [7] :

$$k_{rm}[\nu+1]^2 = k_{rm}[\nu]^2 + (2\nu+1)\left(\frac{2\pi\Delta_f}{c}\right)^2 + \epsilon[\nu], \quad (3)$$

where $\epsilon[\nu] \triangleq k_{zm}[\nu]^2 - k_{zm}[\nu+1]^2$. In shallow-water environments, the vertical wavenumbers k_{zm} weakly depend on the frequency [9]; the quantity ϵ is thus smaller than the other terms of the equation and can be neglected. Dispersive relation has been exploited to track the propagating wavenumbers in the (f-k) diagram in [7] and [8]. We place this work in the continuation of these previous contributions, but in contrast to them, we propose here a method where wavenumbers are not constrained to lie on a discrete grid and that does not require prior knowledge of the modal cut-off frequencies (as in [7]).

3. GRID-FREE COMPRESSED SENSING

For a HLA with L regularly-spaced sensors, Eq. (1) can be rewritten under the following matrix discretized formula for a frequency ν[1]:

$$\mathbf{y}_\nu = \mathbf{D}\mathbf{a}_\nu + \mathbf{n}_\nu, \quad (4)$$

where the elements in $\mathbf{a}_\nu$ are proportional to the modal amplitudes (A_m in Eq. (1)), $\mathbf{n}_\nu$ is an additive noise and $\mathbf{D} \in \mathbb{C}^{L\times N}$ is a dictionary made up of Fourier atoms, that is $\mathbf{d}_{\kappa_{rn}} \triangleq [1, \ldots, e^{-j(L-1)\Delta_r\kappa_{rn}}]^T$, where Δ_r is the inter-sensor distance and the $\{\kappa_{rn}\}_{n\in\{1,\ldots,N\}}$ constitutes a N-point discretization of the possible values for the $k_{rm}[\nu]$. Then performing a (spatial) FT of the received signal $\mathbf{y}_\nu$ basically amounts to a least-square estimation of $\mathbf{a}_\nu$.

Since only a few modes are expected to propagate in the medium, sparse representations (SR) come as a natural and intuitive model for the search over a large (possibly infinite) set of wavenumbers. Under this assumption, $\mathbf{D}$ is seen as an overcomplete dictionary (*i.e.* $L \ll N$) while $\mathbf{a}_\nu$ is assumed to contain many zeros. Integrating the sparsity of $\mathbf{a}_\nu$ into the estimation procedure can be expressed as follows:

$$\hat{\mathbf{a}}_\nu = \underset{\mathbf{a}_\nu}{\text{argmin}} \|\mathbf{y}_\nu - \mathbf{D}\mathbf{a}_\nu\|_2^2, \text{ subject to } \|\mathbf{a}_\nu\|_0 \leq M_\nu, \quad (5)$$

with $\|\mathbf{a}_\nu\|_0$ the ℓ_0 pseudo-norm which counts the numbers of non-zero entries of $\mathbf{a}_\nu$ and M_ν the maximum number of wavenumbers expected to propagate at frequency bin ν.

There are several classical algorithms dealing with the previous equation (see *e.g.* [10] for a review). Among the most popular ones, we can mention orthogonal matching pursuit (OMP, [11]) which builds up the solution of (5) by making a succession of greedy decisions, or basis pursuit (BP, [12]) which replaces the ℓ_0 norm by an ℓ_1 norm leading to a released version of (5) and then resorts to standard optimization procedure. Both of them have been used in the particular case of wavenumbers estimation in [13],[14].

As implied in the matrix formulations (4)–(5), these algorithms classically exploit a discretized grid to recover the non-zero coefficients of $\mathbf{a}_\nu$. This can result in a lack of precision which can be prejudicial for a fine characterization of the underlying physics. These limitations can be somehow alleviated by considering a grid-free setting. Practically, a grid-free version of the relaxation of (5) can be obtained by replacing the ℓ_1 norm (only valid in a finite dimensional setting) by the total variation norm (see *e.g.* [18]). The solution of the sparse reconstruction problem can then be obtained by using the CVX software [15]. This is the approach considered in [16] for the problem of wavenumber estimation with vertical line arrays. A main drawback of such procedure is its high complexity and consequently computational time. Simultaneously, a continuous (or grid-free) version of OMP has been proposed in [17]. The approach makes use of a dictionary augmented with shifted copies of each atom's derivative. Then a continuous sparse representation of $\mathbf{y}$ can be estimated through the formalism of a Taylor development. Following this idea, we propose to resort to another intuitive mathematical operation by adding a gradient-descent step into the classical OMP procedure.

[1]For the sake of clarity we omit the dependence in z since we consider HLA, *i.e.* sensors at constant depth.

4. PROPOSED PROCEDURE

4.1. Grid-free OMP

The grid-free procedure that we propose for OMP is described in Algorithm 1. As such, it can be applied to estimate wavenumbers from each frequency line of the (f-k) diagram independently from each other.

Algorithm 1 Grid-free Orthogonal Matching Pursuit

0. Initialization : $\mathbf{r}_\nu = \mathbf{y}_\nu, \mathcal{S}_\nu = \emptyset$
While stopping criterium is not reached, repeat
1. Find new propagating wavenumber

$$\hat{\kappa}_{rn} = \underset{\kappa_{rn}}{\text{argmax}} \ |\langle \mathbf{r}_\nu, \mathbf{d}_{\kappa_{rn}} \rangle|, \tag{6}$$

where $\mathbf{d}_{\kappa_{rn}}$ is the n-th column of $\mathbf{D}$.
2. Apply gradient-descent algorithm to refine previous estimate and get $\hat{k}_{rm}[\nu]$. Set $\mathcal{S}_\nu = \mathcal{S}_\nu \cup \hat{k}_{rm}[\nu]$.
3. Compute corresponding coefficients

$$\hat{\mathbf{a}}_{\nu,\mathcal{S}_\nu} = \mathbf{D}^+_{\mathcal{S}_\nu} \mathbf{y}_\nu \tag{7}$$

with $\mathbf{D}^+_{\mathcal{S}_\nu}$ the Moore-Penrose pseudo-inverse matrix of $\mathbf{D}_{\mathcal{S}_\nu}$ made up of Fourier atoms specified by $\mathcal{S}_\nu$.
4. Update residual : $\mathbf{r}_\nu = \mathbf{y}_\nu - \mathbf{D}_{\mathcal{S}_\nu} \hat{\mathbf{a}}_{\nu,\mathcal{S}_\nu}$

Algorithm 1 can be stopped when a prescribed number of modes is obtained or when the energy of the residual $\mathbf{r}_\nu$ is low enough. As in practice, the precise number of propagating modes is often unknown, the energy-based stopping criterium can be applied, based on an estimation of the noise power.

4.2. Tracking modes

In the continuation of [8], we propose to additionally integrate the information brought by the dispersion relation (3). In other words, the non-zero components of $\mathbf{a}_\nu$ can no longer be chosen independently from each other but must follow some physics-driven inter-rules when ν is changed. To this end, we put the physical dependency of the modes among time frequencies into the line-by-line processing of the (f-k) diagram.

More precisely, the resulting algorithm starts at the first frequency line and estimates the most correlated atom of $\mathbf{D}$ with respect to the data. If the scalar product between this atom and the data is higher than a specific threshold, the algorithm considers it as a propagating mode and estimates a predicted interval for the wavenumber of this mode for the next frequency. At the next frequency, existing modes are searched in their predicted intervals of the space spectrum using simple gradient-descent procedures. They are retained if their amplitudes are "large enough".

In the targeted applications, mode wavenumbers are well-enough separated and modes possibly appearing at this new frequency can be searched among wavenumbers outside the predicted intervals. Note also that a mode that appeared at a given frequency will continue to exist at higher frequencies. All of these properties can be used to design an adaptative strategy for detection thresholds. Letting T_0 the threshold for detecting a new mode, and $\nu_{c,m}$ the frequency index at which the m-th mode appeared, for $\nu \geq \nu_{c,m}$, we set a threshold $T_{m,\nu} = \lambda^{\nu-\nu_{c,m}} T_0$. The interest of $T_{m,\nu}$ is to prevent us from propagating false detections among frequencies by checking that the amplitudes of the modes found in the predicted wavenumber intervals are not "too low". It is natural to decrease $T_{m,\nu}$ with ν as a mode already detected for a large number of frequency lines will probably represent a true detection. However, adjusting λ may not be straightforward. Alternatively, we propose to choose $T_{m,\nu} = T_0 - \sum_{i=1}^{\nu-\nu_{c,m}} \lambda^i T_0$. Letting T_∞ a desired limit threshold, a straightforward calculation shows that we should have $\lambda = \frac{1-T_\infty/T_0}{2-T_\infty/T_0}$. Then it is interesting to note that $1-2\lambda = (1-\lambda)\frac{T_\infty}{T_0} \geq 0$, hence $\lambda \in [0, 1/2]$. We can choose $\lambda = 1/2$ as it corresponds to the choice $T_\infty = 0$. Alternatively, we can set an asymptotic false alarm probability α_∞ that we accept for propagating modes. Under Gaussian complex circular noise assumption with total variance σ^2, we then get $T_\infty = \sigma\sqrt{-2\log \alpha_\infty}$. Note that, we propose a similar definition for T_0, that is $T_0 = \sigma\sqrt{-2\log\alpha_0}$, where α_0 is a given initial false alarm probability. Algorithm 2 resumes the proposed procedure.

Algorithm 2 Grid-free Orthogonal Matching Pursuit considering dispersive relation

0. Initialization :$\forall \nu \in \{1, .., F\}$, $\mathbf{r}_\nu = \mathbf{y}_\nu$, $\mathcal{S}_\nu = \emptyset$, $I_{0,\nu} = \emptyset$, $M = 0$.
For $\nu = 1 : F$
1. Find new propagating wavenumber

$$\hat{\kappa}_{rn} = \underset{\kappa_{rn}}{\text{argmax}} |\langle \mathbf{r}_\nu, \tilde{\mathbf{d}}_{\kappa_{rn}} \rangle|, \tag{8}$$

where $\tilde{\mathbf{d}}_{\kappa_{rn}}$ is the n-th column of $\mathbf{D}_{\cup_{m\in\{0,\dots,M\}} I_{m,\nu}}$ made up of Fourier atoms not in $\cup_{m\in\{0,\dots,M\}} I_{m,\nu}$.
If $|\langle \mathbf{r}_\nu, \tilde{\mathbf{d}}_{\hat{\kappa}_{rn}} \rangle| \geq T_0$
– Set $M = M + 1$.
– Apply gradient-descent algorithm to refine previous estimate and get $\hat{k}_{rm}[\nu]$. Set $\mathcal{S}_\nu = \mathcal{S}_\nu \cup \hat{k}_{rm}[\nu]$.
– Compute corresponding coefficients and update residual :
$\mathbf{r}_\nu = \mathbf{y}_\nu - \mathbf{D}_{\mathcal{S}_\nu} \hat{\mathbf{a}}_{\nu,\mathcal{S}_\nu}$ with $\hat{\mathbf{a}}_{\nu,\mathcal{S}_\nu} = \mathbf{D}^+_{\mathcal{S}_\nu} \mathbf{y}_\nu$.
2. If $M > 1$, propagate existing modes
For $m = 1 : M - 1$
– Apply gradient-descent on interval $I_{m,\nu}$ and get $\hat{k}_{rm}[\nu]$. If $|\langle \mathbf{r}_\nu, \mathbf{d}_{\hat{k}_{rm}[\nu]} \rangle| \geq T_{m,\nu}$, set $\mathcal{S}_\nu = \mathcal{S}_\nu \cup \hat{k}_{rm}[\nu]$.
– Compute corresponding coefficients and update residual :
$\mathbf{r}_\nu = \mathbf{y}_\nu - \mathbf{D}_{\mathcal{S}_\nu} \hat{\mathbf{a}}_{\nu,\mathcal{S}_\nu}$ with $\hat{\mathbf{a}}_{\nu,\mathcal{S}_\nu} = \mathbf{D}^+_{\mathcal{S}_\nu} \mathbf{y}_\nu$.
3. Predict propagating intervals for next frequency
For all $m \in \{1, \dots, M\}$, define

$$I_{m,\nu+1} = \left[\tilde{k}_{rm}[\nu+1] - \epsilon[\nu]; \tilde{k}_{rm}[\nu+1] + \epsilon[\nu]\right] \tag{9}$$

where $\tilde{k}_{rm}[\nu+1] = \sqrt{\hat{k}_{rm}[\nu]^2 + (2\nu+1)\left(\frac{2\pi\Delta_f}{c}\right)^2}$.

5. EXPERIMENTS

In this section, we present the experiments led on simulated data to assess the performance of the proposed method. The

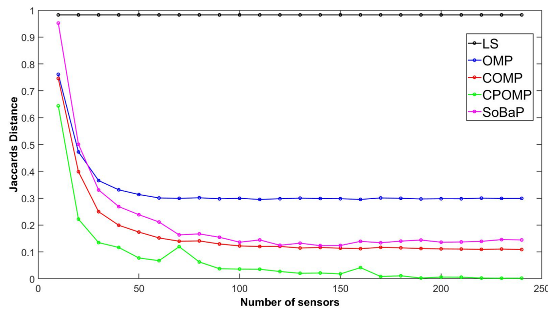

Fig. 2. Impact of sensors number for Jaccard's distance.

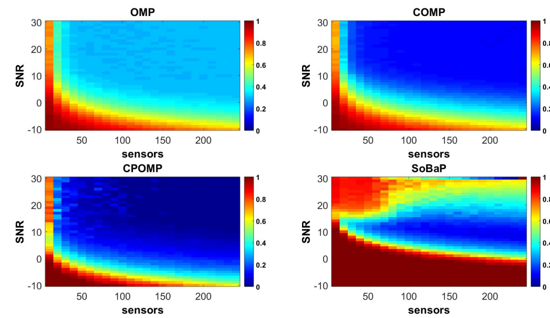

Fig. 3. Impact of SNR and sensors number for Jaccard's distance.

data were simulated by using a Pekeris waveguide [9]. The water column is assumed to be $D = 130$ m deep with a constant sound speed $c_{water} = 1500$ m/s and a density $\rho_{water} = 1$ kg/m^3. The seabed is a semi-finite fluid layer with a sound speed of $c_{seabed} = 2000$ m/s and a density $\rho_{seabed} = 2$ kg/m^3. The sensors array is composed of 240 hydrophones setting on the seabed and regularly spaced of $\Delta_r = 25$ m resulting in a 6000 m long synthetic antenna. The source emits a broadband signal between 0 and 50 Hz. The frequency resolution is $\Delta_f = 0.2$ Hz. The source is placed at the depth of $D = 130$ m to avoid nodes of the modal functions. At 50 Hz we consider that five modes are propagating. We compare 5 algorithms: i) the well-known least-square approach (LS), ii) a basic use of OMP with a stopping criterium considering the number of propagating modes at each frequency (OMP), iii) a continuous version of OMP with the same stopping criterium (COMP), iv) the SoBaP algorithm [8] in its propagated version (SoBaP), and v) the proposed method using the dispersive relation with $\alpha_0 = 0.001$, $\alpha_\infty = 0.5$ and ϵ is arbitrarily set to 10^{-3} (CPOMP). It should be noted that the choice of the number of propagating modes as stopping criterium can be considered as advantageous for OMP and COMP. This is to highlight the good performance of the proposed approach.

We consider two figures of merit: performance of the algorithms is assessed on one hand as a function of the SNR (taken between -10 dB and 30 dB) and on the other, as a function of the number of sensors (randomly chosen among the 240 sensors of the "entire" antenna). Performance is evaluated using the Jaccard's metric [19, 20] which is well-suited to quantitatively assess the detection performance of continuous algorithms. Considering TP the number of wavenumber recovered by the algorithm (according to a tolerance radius R), FN the number of unrecovered wavenumber and FP the number of false alarms, the Jaccard's distance is defined as follow:

$$J_D = 1 - \frac{TP}{TP + FN + FP} \tag{10}$$

According to this formula, a value close to 0 implies a good recovery of the model by the algorithm and at the opposite a value close to 1 corresponds to a lack of recovery or a hudge number of false alarms.

In Fig. 2 (with a SNR set to 12 dB), we can observe the limitations of the LS algorithm, this algorithm builds up a coefficient for each point of the research grid. That means that the number of false alarms is huge and leads to a Jaccard's distance close to 1. We can see the upgrade of the performance by using an algorithm taking into account the sparsity of the problem. This version of OMP stops when it finds the right number of wavenumbers. Nevertheless the performance of this algorithm reaches an asymptote due to the no-recovery of one or more wavenumbers associated to small modal amplitudes. By taking into account the dispersive relation (SoBaP) or the add of an off-grid step (COMP), the performance of the algorithm can be improved. Note that the difference of performance for the SoBaP algorithm is due to the propagation of false alarms. As our algorithm combines both the dispersive relation and the off-grid step, it seems reasonable to see it outperforming the other approaches, explaining the performance of this algorithm over the others for every number of sensors selected. Figure 3 presents a general overview of the performance achieved by the different algorithms for different experimental setups (SNR vs number of sensors). We globally observe a similar behavior as the one illustrated in Fig. 2 on the particular case SNR=12 dB. Hence, OMP (top-left) never seems to achieve perfect detection (Jaccard's distance equal to 0), while COMP (top-right) and CPOMP (bottom-left) present clear transition phases. A more precise comparison between both COMP and CPOMP graphs highlights the contribution of the dispersion relationship in the detection of wavenumbers : the red and light blue zones are much smaller, the dark blue one is much bigger.

6. CONCLUSION

We proposed here a new acoustic array processing to perform high resolution estimation of wavenumbers in shallow-water environments. The method exploits both sparsity and physics in a grid-free setting. Its simple structure (combining OMP and gradient-descent) makes it an intuitive and easy-to-use tool while achieving robust performance in comparison with other (grid-free) sparsity- and/or physics-aware algorithms.

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